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Tabular graphs and chromatic sum

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Abstract

The chromatic sum of a graph is the minimum total of the colors on the vertices taken over all possible proper colorings using positive integers. Erdös et al [Graphs that require many colors to achieve their chromatic sum, Congr. Numer. 71 (1990) 17–28.] considered the question of finding graphs with minimum number of vertices that require t colors beyond their chromatic number k to obtain their chromatic sum. The number of vertices of such graphs is denoted by $P(k,t)$. They presented some upper bounds for this parameter by introducing certain constructions. In this paper we give some lower bounds for $P(k,t)$ and considerably improve the upper bounds by introducing a class of graphs, called tabular graphs. Finally, for fixed t and sufficiently large k the exact value of $P(k,t)$ is determined.

Keywords: Graph coloring; Chromatic sum; Strength

1. Introduction

We consider finite undirected graphs with no loops or multiple edges and refer the reader to [1,2] for notations.

For a graph G, denote by $\sum_k(G)$ the smallest possible sum among all proper k-colorings of G using positive integers. It is not hard to see that $\sum_k(G)$ is not increasing in k. The *chromatic sum* of G, denoted by $\sum(G)$, is defined as $\min_k \sum_k(G)$ and the *strength* of G,

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denoted by $s(G)$, is the smallest number s such that $\sum_s(G)=\sum(G)$. Evidently, $s(G)\geqslant\chi(G)$ for every graph G, but perhaps surprisingly equality does not always hold. In fact, it is shown in [7] for every positive integer k that the strength of almost all trees is greater than k [7]. For a given coloring $c: V(G)\longrightarrow N$ and a positive integer $i\in N$, we denote $c^{-1}(i)$ by C_i and $\sum_i i|C_i|$ by $\sum(G,c)$. A coloring c is called *minimal coloring* of G if for every vertex $v\in V(G)$, $c(v)\leqslant s(G)$ and $\sum(G,c)=\sum(G)$.

For a minimal coloring c the following results are immediate consequences of the above definitions.

($\mathbf{S}_1$) $|C_1|\geqslant|C_2|\geqslant\cdots\geqslant|C_s|$.
($\mathbf{S}_2$) $\sum(G\backslash C_1)=\sum(G)-|V(G)|$.
($\mathbf{S}_3$) $s(G\backslash C_1)=s(G)-1$.
($\mathbf{S}_4$) C_i is a maximal independent set in $G\backslash\bigcup_{j<i}C_j$.
($\mathbf{S}_5$) For every graph G we have $\sum_{\chi(G)}(G)\leqslant\lfloor\frac{\chi(G)+1}{2}|V(G)|\rfloor$.

Also, one can easily prove the following lemma.

Lemma 1. *Let G be a graph and c be a t-coloring of G for which $|C_1|\geqslant\cdots\geqslant|C_t|$ and let $|V(G)|=q|C_1|+r,\ 0\leqslant r<|C_1|$. Then $\sum(G,c)\geqslant\frac{1}{2}q(q+1)|C_1|+r(q+1)$.*

The notion of chromatic sum was first introduced by Kubicka [6]. In [7], Kubicka and Schwenk showed that for every natural number $s\geqslant 2$, there is a tree with strength s having t_s vertices, where

$$t_s := \frac{1}{\sqrt{2}}((2+\sqrt{2})^{s-1}-(2-\sqrt{2})^{s-1}).$$

In 1989, Thomassen et al. [8] obtained interesting tight bounds for the chromatic sum which depend only on the number of edges. Also, Hajiabolhassan et al. [4] obtained the following bound for the strength of graphs:

$$s(G)\leqslant\left\lceil\frac{col(G)+\Delta(G)}{2}\right\rceil,$$

where $col(G)$ and $\Delta(G)$ denote *coloring number* and *maximum degree* of graph G, respectively. Moreover, Jiang and West [5] showed that this upper bound is sharp. The following conjecture is suggested in [4].

Conjecture A. For every graph G:

$$s(G)\leqslant\left\lceil\frac{\chi(G)+\Delta(G)}{2}\right\rceil,$$

where $\chi(G)$ is the chromatic number of graph G.

For every integer $k\geqslant 2$ and positive integer t, let $\mathscr{G}_k^t$ denote the family of all graphs G with $\chi(G)=k$ and $s(G)\geqslant k+t$. In [3] Erdös et al. provided three methods to construct certain graphs in $\mathscr{G}_k^t$. They used the notation $P(k,t)$ for the minimum number of vertices among all graphs in $\mathscr{G}_k^t$. They also defined $\pi(t)$ as $\min_k P(k,t)$. The following results appeared in [3].

Theorem A. *We have* $P(2,1)=P(3,1)=8$ *and for every integer* $k \geqslant 4$,

$$P(k,1)=k+1+\left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor.$$

By this theorem $\pi(1)=8$.

Corollary A. *Let* $G \in \mathscr{G}_k^1$ *and let c be a minimal coloring of G. Then*

$$|C_1| \geqslant 1+\left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor.$$

Also, based on some constructions they proved the following results.

Theorem B (*Erdös et al. [3]*). *For every pair of integers* $k \geqslant 2$ *and* $t \geqslant 1$,

$$P(k,t) \leqslant (k(k-1)+1)^{\lfloor \frac{t}{k-1}-1 \rfloor}(kt_{\mathrm{mod}\,(k-1)}+1)k^3.$$

Corollary B. *For every positive integer t*, $P(2,t) \leqslant 8 \cdot 3^{t-1}$.

Corollary C. *For every positive integer t*, $\pi(t) \leqslant (t+1)^3$.

By introducing a new class of graphs, called tabular graphs, we improve the bounds given in Theorem B and Corollary C. Tabular graphs are introduced and studied in Section 2. In Section 3, we give some upper bounds and by estimating the size of color classes in minimal colorings, we obtain two different lower bounds for $P(k,t)$. In Section 4 we give the exact values of $P(2,t)$, $P(3,t)$, and $P(k,2)$. Also, for fixed t and sufficiently large k we obtain an explicit formula for $P(k,t)$.

2. Tabular graphs

In this section, we introduce and study a new class of graphs called tabular graphs, which plays an important role in this paper. In fact, we show that for computing $P(k,t)$ it is sufficient to consider only these graphs. On the other hand, tabular graphs are useful in other areas of graph theory, especially in coloring problems.

Let $\{V_{ij}\}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, be mn mutually disjoint finite sets. A *table graph* $T[V_{ij}]$ is a graph whose vertex set is $\bigcup_{i,j} V_{ij}$ and two vertices $u \in V_{ij}$ and $v \in V_{kl}$ are adjacent if and only if $i \neq k$ and $j \neq l$. A graph G is *tabular* if it is isomorphic to a table graph.

Clearly, a table graph $T[V_{ij}]$ can be represented (up to isomorphism) by a table whose (i,j) entry is $|V_{ij}|$. For example the complete graph K_n can be represented by an $n \times n$

(a)

2	0
0	2

(b)

1	1	0
0	0	2

Fig. 1. Two tables representing C_4.

table graph whose (i, j) entry is δ_{ij}, where δ_{ij} is the Kronecker delta. A tabular graph may be represented by several different tables as shown in Fig. 1.

We use some terminologies for abbreviation. We call each V_{ij} a *cell*, $\bigcup_j V_{ij}$ a *row* and $\bigcup_i V_{ij}$ a *column* of $T[V_{ij}]$. Also, by a *line* we mean a row or a column. We say that two vertices are *equivalent* if they belong to the same cell. Two vertices in a line are called *colinear*.

The following statements on table graphs and tabular graphs are immediate consequences of the above definitions.

($\mathbf{T}_1$) Every induced subgraph of a tabular graph is tabular.
($\mathbf{T}_2$) In a table graph two vertices are independent if and only if they are colinear.
($\mathbf{T}_3$) In a table graph a set of vertices in which no two are colinear forms a clique.
($\mathbf{T}_4$) A maximal independent set of vertices in a table graph is always a line. Conversely, a line with at least two non-equivalent vertices is a maximal independent set. Notice that a line with only one non-empty cell, say the first column in Fig. 1(b), is not necessarily a maximal independent set.

By ($\mathbf{T}_4$) the chromatic number of a table graph is equal to the minimum number of lines which cover all the non-zero entries of its table. Also, by ($\mathbf{T}_3$) the clique number of a table graph is the maximum number of mutually non-colinear-vertices of its table. Hence, by the well-known König's lemma, for every tabular graph G we have $\chi(G) = \omega(G)$, and by ($\mathbf{T}_1$) the following proposition follows.

Proposition 1. *Every tabular graph is perfect.*

In what follows we show the motivation for using tabular graphs in the study of minimal colorings and chromatic sum. First notice that determining the chromatic sum of a graph deal with numerical calculations and tabular graphs can be represented by a table of numbers.

Proposition 2. *Every graph G in $\mathscr{G}_k^t$ which has a minimum number of vertices and a maximum number of edges is a tabular graph.*

Proof. Let c be a minimal coloring of G and let c' be a k-coloring of G. Set $V_{ij} = C_i \cap C'_j$. Obviously, if $u \in V_{ij}$ and $v \in V_{kl}$ are adjacent, then $i \neq k$ and $j \neq l$ and by the maximality condition on the number of edges, two such vertices are adjacent. Therefore, G is a tabular graph. □

Proposition 2 shows why the application of tabular graphs is useful in calculating $P(k, t)$.

Lemma 2. *Let $G = T[V_{ij}]$ be a table graph and let c be a minimal coloring of G. Then, the following assertions hold*:

(i) *If C_i contains two non-equivalent vertices of a line L, then $C_i = L \setminus \bigcup_{j<i} C_j$. In particular, in every minimal coloring, equivalent vertices lie in the same color class.*
(ii) *If R_k and R_l are two rows such that $|V_{kj}| \leqslant |V_{lj}|$ for all j's and $|V_{kj_0}| < |V_{lj_0}|$ for some j_0, then $C_1 \neq R_k$.*

Proof. (i) By ($\mathbf{S}_4$) and ($\mathbf{T}_4$) this part is clear.

(ii) Suppose on the contrary that $C_1 = R_k$. We define a new coloring c' by the following changes in coloring c. Set $c'(R_l) = \{1\}$ and $c'(V_{kj}) = c(V_{lj})$, for all j. We claim that c' is actually a proper coloring. Suppose v is a vertex of G for which $c'(v) = c(v)$. If $c'(v) = c'(V_{kj})$ then $c(v) = c(V_{lj})$ and so v and the vertices of V_{lj} are colinear. Since $v \notin R_l$, v thus belongs to the jth column and therefore v and the vertices of V_{kj} are non-adjacent. Thus, c' is a proper coloring. Obviously, $\sum(G, c') < \sum(G, c) = \sum(G)$, so c is not a minimal coloring, a contradiction. □

The above lemma shows that a minimal coloring of a table graph is not complicated and as we will see later for many table graphs that the set of color classes of a minimal coloring is the set of rows or the set of columns.

For a table graph $G = T[V_{ij}]$, by the *row* (resp. *column*) *coloring* we mean a coloring c for which $C_i = \bigcup_j V_{ij}$ (resp. $\bigcup_j V_{ji}$). We denote the row coloring by $\mathscr{R}$ and the column coloring by $\mathscr{C}$.

Now by Lemma 1 the following corollary holds.

Corollary 1. *Let G be a table graph whose columns all have the same size. If in a minimal coloring c of G, C_1 is one of the columns, then $\sum(G) = \sum(G, \mathscr{C})$.*

As an application of Lemma 2 we prove for $k \geqslant 4$, $P(k, 1) \leqslant k + 1 + \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$. Let $a = 1 + \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$. It is easy to verify that for $k \geqslant 4$ we have $a \leqslant k$. Now consider the $(k+1) \times k$ table graph $Q = T[V_{ij}]$ defined by

$$|V_{ij}| = \begin{cases} 1 & \text{if } i = 1 \text{ and } 1 \leqslant j \leqslant a, \\ \delta_{i,j+1} & \text{otherwise.} \end{cases}$$

This is the graph constructed by Erdös et al. [3] by which they proved Theorem 4 for $k \geqslant 4$.

We claim that in every minimal coloring c of Q, C_1 is the first row. Otherwise, by Lemma 2(i), the set C_1 must be a column and so for all i's, $|C_i| \leqslant 2$. Also, one can observe that there are $|V(Q)| - 2a$ vertices in Q which have degree $|V(Q)| - 1$ and so in every coloring of Q, each of these vertices forms a singleton color class. Thus, there are at most a color classes of size 2. On the other hand, the column coloring of Q has exactly a such color classes. Hence, by the above discussion we have $\sum(Q, c) \geqslant \sum(Q, \mathscr{C})$ and by a simple computation it can be shown that $\sum(Q, \mathscr{R}) < \sum(Q, \mathscr{C})$, a contradiction. Thus, C_1 is the first row and $Q \in \mathscr{G}_k^1$.

3. Upper and lower bounds for $P(k, t)$

In this section we first give an upper bound for $P(k,t)$ which is based on the construction of table graphs.

Theorem 1. *For every pair of integers $k \geqslant 4$ and $t \geqslant 1$,*

$$P(k,t) \leqslant \frac{k(k+3)}{2}\left(\frac{k+1}{k-1}\right)^{t-1} - \binom{k}{2}.$$

Proof. For $k \geqslant 4$, let $Q_{k,0}$ be the table representing the complete graph K_k, and construct $Q_{k,t}$ by adding a new first row to $Q_{k,t-1}$ whose cells all have the size f_t, where

$$f_t = \left\lceil \frac{2}{k(k-1)} |V(Q_{k,t-1})| \right\rceil.$$

Here the table graph has k columns and $k+t$ rows. We will prove by induction that in every minimal coloring c of $Q_{k,t}$, C_1 is the first row and $\sum(Q_{k,t}) < \sum(Q_{k,t}, \mathscr{C})$. Since $k \geqslant 4$ the assertion is true for $t = 1$. For $t \geqslant 2$, suppose on the contrary that C_1 is a column. By Corollary 1, $\sum(Q_{k,t}) = \sum(Q_{k,t}, \mathscr{C}) = \frac{1}{2}k(k+1)f_t + \sum(Q_{k,t-1}, \mathscr{C})$. But,

$$\sum(Q_{k,t}) \leqslant \sum(Q_{k,t}, \mathscr{R}) = kf_t + \sum(Q_{k,t-1}) + |V(Q_{k,t-1})| < \sum(Q_{k,t}, \mathscr{C}).$$

Note that the last inequality is a consequence of the induction hypothesis and the choice of f_t. Hence, by Lemma 2(ii) C_1 must be the first row and therefore, $Q_{k,t} \in \mathscr{G}_k^t$.

Next, we have

$$|V(Q_{k,t})| = kf_t + |V(Q_{k,t-1})| \leqslant \frac{k+1}{k-1}|V(Q_{k,t-1})| + k.$$

Using q_t for $|V(Q_{k,t})| + \binom{k}{2}$ we can rewrite the above inequality as $q_t \leqslant \frac{k+1}{k-1} q_{t-1}$. It follows that

$$P(k,t) \leqslant |V(Q_{k,t})| \leqslant \left(\binom{k}{2} + 2k\right)\left(\frac{k+1}{k-1}\right)^{t-1} - \binom{k}{2}. \qquad \square$$

Corollary 2. *For every integer $t \geqslant 4$, $\pi(t) \leqslant \frac{e^2-1}{2}t^2 + \frac{3e^2+1}{2}t$, where e is the Napier number.*

Proof. Take $k = t$. Using the inequality $\left(\frac{t+1}{t-1}\right)^{t-1} \leqslant e^2$ and Theorem 1, we obtain

$$\pi(t) \leqslant \frac{e^2}{2}t(t+3) - \binom{t}{2} = \frac{e^2-1}{2}t^2 + \frac{3e^2+1}{2}t. \qquad \square$$

Remark 1. By Theorem A, $\pi(1) = 8$ and we will show later that $\pi(2) = 14$ and $\pi(3) \leqslant 30$. Thus, the above bound for $\pi(t)$ is valid for every positive integer t.

In the rest of this section we give some lower bounds for $P(k,t)$.

Lemma 3. *Suppose that* $v=\sum_{i=1}^{n} p_i=\sum_{j=1}^{m} q_j$, $n \geqslant m$, *and* $\sum_{i=1}^{n} ip_i < \sum_{j=1}^{m} jq_j$, *where* $p_1 \geqslant \cdots \geqslant p_n$, *and* $q_1 \geqslant \cdots \geqslant q_m$ *are non-negative rational numbers. Then* $p_1 > \frac{v}{m}$.

Proof. For every $1 \leqslant j \leqslant \lceil \frac{m}{2} \rceil$, we have $jq_j + (m-j+1)q_{m-j+1} \leqslant \frac{m+1}{2}(q_j + q_{m-j+1})$. Thus, $\sum_{j=1}^{m} jq_j \leqslant \frac{(m+1)v}{2}$. Now if $p_1 \leqslant \frac{v}{m}$, then the minimum value of $\sum_{i=1}^{n} ip_i$ is obtained when $p_1 = \cdots = p_m = \frac{v}{m}$ and $p_{m+1} = \cdots = p_n = 0$. Hence,

$$\sum_{i=1}^{n} ip_i \geqslant \sum_{i=1}^{m} i\frac{v}{m} = \frac{(m+1)v}{2} \geqslant \sum_{j=1}^{m} jq_j,$$

a contradiction. $\square$

The following two propositions are used in the proof of Theorem 2.

Proposition 3. *Let* $k \geqslant 2$ *and* $t \geqslant 1$ *be positive integers. Then we have* $P(k+1, t) \leqslant P(k, t+1)$.

Proof. Suppose that G is in $\mathscr{G}_k^{t+1}$ and $|V(G)| = P(k, t+1)$. Let c be a minimal coloring of G. Now by adding sufficient edges to graph G between color classes of c one can construct a graph G^* such that $\chi(G^*) = k+1$. Thus, $G^* \in \mathscr{G}_{k+1}^{t}$ and this implies that $P(k+1, t) \leqslant |V(G^*)| = |V(G)| = P(k, t+1)$, as desired. $\square$

Proposition 4. *Let* G *be a graph with* $s(G) > \chi(G)$ *and let* c *be a minimal coloring of* G. *Then we have*

(i) $|C_1| > \dfrac{|V(G)|}{\chi(G)}$.

(ii) $|C_1| \geqslant 1 + \left\lfloor \dfrac{1+\sqrt{8(s(G)-1)+1}}{2} \right\rfloor$.

Proof. (i) Assume that c' is a $\chi(G)$-coloring of G, $p_i = |C_i|$, $1 \leqslant i \leqslant s(G)$, and $q_j = |C'_j|$, $1 \leqslant j \leqslant \chi(G)$. Now Lemma 3 implies that (i) holds.

(ii) By adding sufficient edges to graph G between color classes of c, one can construct a graph G^*, such that $\chi(G^*) = s(G) - 1$ and $s(G^*) = s(G)$. So $G^* \in \mathscr{G}_{s(G)-1}^{1}$ and Corollary A implies the assertion. $\square$

Now we are ready to present the lower bounds.

Theorem 2. *For every pair of integers* $k \geqslant 2$ *and* $t \geqslant 1$, *the following inequalities hold*:

(i) $P(k, t) \geqslant \left(\dfrac{k}{k-1}\right)^{t-1} P(k, 1)$.

(ii) $P(k, t) \geqslant k + t + t\left\lfloor \dfrac{1+\sqrt{8k+1}}{2} \right\rfloor$.

Proof. Let G be in $\mathscr{G}_k^t$ and c be a minimal coloring of G.

(i) Consider the graph $H = G - C_1$. If $\chi(H) = k$, then $|V(H)| \geqslant P(k, t-1)$, otherwise $|V(H)| \geqslant P(k-1, t)$. But by Proposition 3, $P(k-1, t) \geqslant P(k, t-1)$ and so in any case we have $|V(H)| \geqslant P(k, t-1)$. On the other hand, by Proposition 4(i) we have $|C_1| > \frac{|V(G)|}{k}$ which yields the inequality $|C_1| \geqslant \frac{1}{k-1}|V(H)|$. Thus, we obtain $|V(G)| \geqslant \frac{k}{k-1}|V(H)|$, and in particular,

$$P(k, t) \geqslant \frac{k}{k-1} P(k, t-1),$$

which immediately implies (i).

(ii) For $j \geqslant k+1$, by ($\mathbf{S}_3$), the subgraph $G[\bigcup_{i=s-j+1}^{s} C_i]$, $j \geqslant k+1$, has strength j and the coloring $c'(v) = c(v) - s + j$ is a minimal coloring for it. Hence, by Proposition 4(ii) it is obtained that $|C_j| \geqslant 1 + \left\lfloor \frac{1+\sqrt{8(j-1)+1}}{2} \right\rfloor$. Summing up over all $j \geqslant k+1$ and noting that $|\bigcup_{i=t+1}^{k+t} C_i| \geqslant k$ we obtain (ii). $\square$

4. Some exact values of *P*(*k*, *t*)

This section contains some statements about the exact value of $P(k, t)$. First, for fixed t and sufficiently large k we give an explicit formula for $P(k, t)$.

Theorem 3. *For every integer $t \geqslant 1$, there exists a number K such that for each $k \geqslant K$ we have*

$$P(k, t) = k + t + t \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor. \qquad (*)$$

Proof. Denote the right-hand side of the above formula by $q(k, t)$ and suppose $a = 1 + \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$. Assume that K is so large that if $k \geqslant K$ then $k/t \geqslant 1 + \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$. By Theorem 2(ii), it suffices to show that for t given and $k \geqslant t + t \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$ there exists a graph with chromatic number k and strength $k + t$ which has $q(k, t)$ vertices. Due to the condition $k/t \geqslant a$ we construct a $(k+t) \times k$ table graph $G = T[V_{ij}]$ as follows:

$$|V_{ij}| = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant t \text{ and } (i-1)a + 1 \leqslant j \leqslant ia, \\ \delta_{i,j+t} & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\sum(G, \mathscr{R}) < \sum(G, \mathscr{C})$. Now using this fact and by induction we prove that the row coloring of G is a minimal coloring. If $t = 1$ then by Theorem A there is nothing to prove. Let $t \geqslant 2$ and c be a minimal coloring of G. If C_1 is not a row then it must be a column and so for all i's, $|C_i| \leqslant 2$. Also, one can observe that there are $|V(G)| - 2at$ vertices in G which have degree $|V(G)| - 1$ and so in every coloring of G, each of these vertices forms a singleton color class. Thus, there are at most ta color classes of size 2. On the other hand, the column coloring of G has exactly ta such color classes. Hence, by the

$8 \cdot 3^{t\text{-}2}$	$8 \cdot 3^{t\text{-}2}$
$\vdots$	$\vdots$
72	72
24	24
8	8
3	3
1	0
0	1

(a)

$5 \cdot 2^{t\text{-}3}$	$5 \cdot 2^{t\text{-}3}$	$5 \cdot 2^{t\text{-}3}$
$\vdots$	$\vdots$	$\vdots$
5	5	5
2	2	3
2	2	1
1	0	0
0	1	0
0	0	1

(b)

Fig. 2. Tabular graphs for $P(2,t)$ and $P(3,t)$.

above discussion we have $\sum(G,c) \geqslant \sum(G,\mathscr{C}) > \sum(G,\mathscr{R})$, a contradiction. Hence, C_1 is the first row. Now $k/(t-1) > 1 + \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$, and by induction we are done. $\square$

Remark 2. From now on we denote by $\kappa(t)$ the minimum number K, such that $k \geqslant K$ implies $(*)$. By a simple computation one can show that $\kappa(t) \leqslant 2t^2 + 3t$. As we will see in Theorem 6, by modifying the construction used in Theorem 3, it can be shown that the above estimation is far from the real value of $\kappa(t)$.

By using various techniques developed in this paper one can compute the exact value of $P(k,t)$ in some special cases. In the next three theorems we determine the exact values of $P(2,t)$, $P(3,t)$, and $P(k,2)$.

Theorem 4. *For $t \geqslant 1$ we have*

$$P(2,t) = 8 \cdot 3^{t-1}.$$

Proof. By Corollary B, or from the table graph shown in Fig. 2(a), we know that $P(2,t) \leqslant 8 \cdot 3^{t-1}$. Let G be a graph in $\mathscr{G}_2^t$ and let c be a minimal coloring of G. Since G is bipartite and $s(G) > 2$ we have $\sum(G) < \sum_2(G)$; so by $(\mathbf{S}_5)$

$$\sum(G) < \left\lfloor \frac{3|V(G)|}{2} \right\rfloor.$$

Hence, $2(1 + \sum_{i=1}^{t+2} i|C_i|) \leqslant 3|V(G)| = 3\sum_{i=1}^{t+2}|C_i|$, which implies that for $t \geqslant 1$,

$$|C_1| \geqslant \sum_{i=1}^{t+1} (2i-1)|C_{i+1}| + 2.$$

We prove by induction that $|C_1| \geqslant 16 \cdot 3^{t-2}$. For $t=2$, $(\mathbf{S}_3)$ and the above inequality imply that $|C_1| \geqslant 16$. Using $(\mathbf{S}_3)$ and by induction $|C_{i+1}| \geqslant 16 \cdot 3^{t-i-2}$, for $1 \leqslant i \leqslant t-2$. Now $|C_t \cup C_{t+1} \cup C_{t+2}| \geqslant P(2,1) = 8$ and the worst case is when $|C_t| = 6$, and $|C_{t+1}| = |C_{t+2}| = 1$.

Therefore,

$$|C_1| \geqslant 16\left(\sum_{i=1}^{t-2}(2i-1)\cdot 3^{t-i-2}\right)+6(2t-3)+(2t-1)+(2t+1)+2$$
$$=16\cdot 3^{t-2}. \qquad \square$$

We know from Theorem A that $P(3,1)=8$. The following theorem gives the value of $P(3,t)$ for $t\geqslant 2$.

Theorem 5. *For $t\geqslant 2$ we have*

$$P(3,t)=15\cdot 2^{t-2}.$$

Proof. Consider the table graph H indicated in Fig. 2(b). By Corollary 1, one can easily verify that the row coloring of H is a minimal coloring and hence $P(3,t)\leqslant 15\cdot 2^{t-2}$, for $t\geqslant 2$. Let G be a graph in $\mathscr{G}_3^t$, $t\geqslant 2$, and let c be a minimal coloring of G. By $(\mathbf{S}_5)$, $\sum(G)<2|V(G)|$. Hence,

$$|C_1|>\sum_{i=1}^{t+1} i|C_{i+2}|.$$

If $t=2$, $(\mathbf{S}_3)$ and the above inequality imply that $|C_1|\geqslant 7$. Now if $t\geqslant 3$, by induction we will prove that $|C_1|\geqslant 15\cdot 2^{t-3}$. For $t=3$, $(\mathbf{S}_3)$ and the above inequality imply that $|C_1|\geqslant 15$.

If $t\geqslant 4$ then by induction, for $1\leqslant i\leqslant t-4$ we have $|C_{i+2}|\geqslant 15\cdot 2^{t-i-4}$. Now, $|C_{t-1}\cup C_t\cup C_{t+1}\cup C_{t+2}\cup C_{t+3}|\geqslant P(3,2)$; therefore, $|C_{t-1}|\geqslant 7$. Also, $|C_t\cup C_{t+1}\cup C_{t+2}\cup C_{t+3}|\geqslant P(3,1)=8$, so the worst case is when $|C_{t-1}|=7$, $|C_t|=5$, and $|C_{t+1}|=|C_{t+2}|=|C_{t+3}|=1$. Hence

$$|C_1|\geqslant 15\left(\sum_{i=1}^{t-4} i2^{t-i-4}\right)+7(t-3)+5(t-2)+(t-1)+t+(t+1)+1$$
$$=15\cdot 2^{t-3}. \qquad \square$$

Theorem 6. *For $k\geqslant 6$ we have*

$$P(k,2)=k+2+2\left\lfloor\frac{1+\sqrt{8k+1}}{2}\right\rfloor.$$

Also, $P(2,2)=24$, $P(3,2)=15$, *and* $P(4,2)=P(5,2)=14$.

Proof. By Theorems 4 and 5 the only special cases that we must verify are $k=4$ and $k=5$. The table graphs indicated in Fig. 3 show that both $P(4,2)$ and $P(5,2)$ are at most 14. Let G be a graph in $\mathscr{G}_4^2$ and let c be a minimal coloring of G. By $(\mathbf{S}_5)$ we have $\sum(G)<\frac{5}{2}|V(G)|$.

2	2	1	1
1	1	1	1
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

1	1	1	1	1
1	1	1	1	0
1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

Fig. 3. Tabular graphs for $P(4, 2)$ and $P(5, 2)$.

Hence,

$$3|C_1| + |C_2| > |C_3| + 3|C_4| + 5|C_5| + 7|C_6|,$$

which implies that $|C_1| \geqslant 5$. If $\chi(G - C_1) = 3$ then $G - C_1 \in \mathscr{G}_3^2$ and therefore $|V(G - C_1)| \geqslant 15$, a contradiction. So we have $\chi(G - C_1) = 4$ and Corollary A implies that $|C_2| \geqslant 4$. If $|V(G)| = 13$ then $|C_1| = 5$ and $|C_2| = 4$ and therefore $\sum(G) = 31$, while for an arbitrary 4-coloring φ of G we have $\sum(G, \varphi) \leqslant 4(1) + 3(2) + 3(3) + 3(4) = 31$, a contradiction. Hence, $P(4, 2) = 14$.

For $P(5, 2)$ we use Proposition 4(ii) to obtain $|C_1| \geqslant 5$. As in the previous case we have $\chi(G - C_1) = 5$ and so $|C_2| \geqslant 4$. Therefore, $P(5, 2) \geqslant 14$, as desired.

To complete the proof we must show that $\kappa(2) = 6$. The inequality $k \geqslant t + t \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$, established in the proof of Theorem 3, implies $\kappa(2) \leqslant 12$ and by Theorem 2(ii) it is sufficient to construct a graph in $\mathscr{G}_k^2$ having $k + 2 + 2 \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$ vertices, for $6 \leqslant k \leqslant 11$. For such a k, define a $(k + 2) \times k$ table graph $T_k = T[V_{ij}]$ by

$$|V_{ij}| = \begin{cases} 1 & \text{if } i = 1 \text{ and } 1 \leqslant j \leqslant a_k, \\ 1 & \text{if } i = 2 \text{ and } k - a_k + 1 \leqslant j \leqslant k, \\ \delta_{i,j+2} & \text{otherwise,} \end{cases}$$

where $a_k = 1 + \left\lfloor \frac{1+\sqrt{8k+1}}{2} \right\rfloor$. By a simple computation one can see that the row coloring of T_k is a minimal coloring and therefore $T_k \in \mathscr{G}_k^2$. $\square$

Based on our results it seems that, for fixed k, $P(k, t)$ as a function of t has exponential behavior. The following conjecture formulates a stronger assertion.

Conjecture 1. We have $\lim_{t\to\infty} \frac{P(k,t+1)}{P(k,t)} = \frac{k+1}{k-1}$.

As Theorem 3 shows, for fixed t, if $k \geqslant \kappa(t)$ then $P(k, t)$ is an increasing function in k. Also, our observations show that for $t > 1$ and small values of k, $P(k, t)$ is decreasing in k.

Conjecture 2. The sequence $\{P(k,t)\}_{k=2}^{\infty}$ is co-unimodal, i.e. there exists an integer l_t such that

$$P(2,t) \geqslant P(3,t) \geqslant \cdots \geqslant P(l_t,t) \leqslant P(l_t+1,t) \leqslant \cdots .$$

Finally, the following problem naturally comes to mind.

Problem. Determine the value of $\kappa(t)$ or give a good estimation for it.

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